Matices :-

Definition :- A system of mn numbers arranged in a rectangular formation along m rows & n columns & bounded by brackets [ ] is called an m by n matrix. which is written as m×n matrix. & denoted by capital letter.

Thus $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots & & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$ is matrix of order mn.

$= [a_{ij}]$. , i = i^{th} row & $j = j^{th}$ column.

Special Matrices :-

1. Row & Column matrices.
2. Square matrices
3. Diagonal matrices.
4. Unit matrix.
5. Null matrix.
6. Symmetric & skew symmetric matrices.
7. Triangular matrix.

Matrices Operations :-

1. Equality of matrices: Two matrices are equal if
   1. They are of same order.
   2. each element equal to corresponding element.
2. Addition & substraction of matrices :- each element is add with corresponding element.
3. multiplication of matrix by a scalar:

$$k\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \end{bmatrix}$$

scalar multiplicat^n holds distributive law:
i.e $k(A+B) = kA + kB$.

Example :- Find x, y, z & w given that.

$$3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 5 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 6 & x+y \\ z+w & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+6 & 5+x+y \\ -1+z+w & 2w+5 \end{bmatrix}$$

equating the corresponding elements, we get.

$\therefore\ 3x = x+6 \Rightarrow 2x = 6 \Rightarrow \boxed{x = 3}$

$3y = 5+x+y \Rightarrow 2y = 5+x \Rightarrow \boxed{y = 4}$

$3z = -1+z+w \Rightarrow 2z = w-1$ as $w=5 \Rightarrow \boxed{z = 2}$

$3w = 2w+5 \Rightarrow \boxed{w = 5}$

(4) Multiplication of Matrices :-

Two matrices can be multiplied only when the number of columns in the first equal to the number of rows in second.

i.e the product: $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix} \times \begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \\ n_1 & n_2 \end{bmatrix}$

Now here No. of col^{ms} in 1^{st} = No. of rows in 2^{nd}.

$\therefore$ multiplication is,

$$\begin{bmatrix} a_1l_1+b_1m_1+c_1n_1 & a_1l_2+b_1m_2+c_1n_2 \\ a_2l_1+b_2m_1+c_2n_1 & a_2l_2+b_2m_2+c_2n_2 \\ a_3l_1+b_3m_1+c_3n_1 & a_3l_2+b_3m_2+c_3n_2 \\ a_4l_1+b_4m_1+c_4n_1 & a_4l_2+b_4m_2+c_4n_2 \end{bmatrix}_{4\times 2}$$

in general, if order of ~~m×n~~ 1st matrix is $m\times n$ & order of second matrix is $n\times p$.

then its multiplicat^n matrix is of order $\underline{m\times p}$

Example :- If $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix}$, find the matrix B st $AB = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix}$.

$\Rightarrow$ Let. $AB = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix} \begin{bmatrix} l & m & n \\ p & q & r \\ u & v & w \end{bmatrix} = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 3l+2p+2u & 3m+2q+2v & 3n+2r+2w \\ l+3p+u & m+3q+v & n+3r+w \\ 5l+3p+4u & 5m+3q+4v & 5n+3r+4w \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix}$$

Now compare respective elements, we get.

$3l+2p+2u=3,\ l+3p+u=1,\ 5l+3p+4u=5$ — 1

$3m+2q+2v=4,\ m+3q+v=6,\ 5m+3q+4v=6$ — 2

$3n+2r+2w=2,\ n+3r+w=1,\ 5n+3r+4w=4$ — 3.

solving eq^n 1 by cramers rule,

$$D=\begin{vmatrix}3&2&2\\1&3&1\\5&3&4\end{vmatrix}=3(12-3)-2(4-5)+2(3-15)$$
$$=3(9)-2(-1)+2(-12)$$
$$=27+2-24$$
$$=5$$

$$Dl=\begin{vmatrix}3&2&2\\1&3&1\\5&5&4\end{vmatrix}=5 \Rightarrow l=\frac{Dl}{D}=\frac{5}{5}=1$$
$$\Rightarrow \boxed{l=1}$$

$$Dp=\begin{vmatrix}3&3&2\\1&1&1\\5&5&4\end{vmatrix}=0\ (\because \text{two columns are equal}).$$

$$Du=\begin{vmatrix}3&2&3\\1&3&1\\5&3&5\end{vmatrix}=0\ (\because \text{two columns are equal}).$$

$\therefore \boxed{p=0}$ &

$\boxed{u=0}$

Similarly solving eq^n 2

$$D=\begin{vmatrix}3&2&2\\1&3&1\\5&3&4\end{vmatrix}=5$$

$$Dm=\begin{vmatrix}4&2&2\\6&3&1\\6&3&4\end{vmatrix}=\cancel{4(12-3)-2(}=2\begin{vmatrix}2&2&2\\3&3&1\\3&3&4\end{vmatrix}=0$$

$\therefore m=\dfrac{Dm}{D}=0$

$\therefore \boxed{m=0}$

$Dq=0$ & $Dv=0 \Rightarrow \boxed{q=0}$ & $\boxed{v=0}$

For eq^n 3.

it gives, $x=0, z=0, w=1$.

$$\therefore B=\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}$$

Example: Prove that $A^3-4A^2-3A+11I=0$

where $A=\begin{bmatrix}1&3&2\\2&0&-1\\1&2&3\end{bmatrix}$.

$\Rightarrow$

$$A^2=A\times A=\begin{bmatrix}1&3&2\\2&0&-1\\1&2&3\end{bmatrix}\cdot\begin{bmatrix}1&3&2\\2&0&-1\\1&2&3\end{bmatrix}=\begin{bmatrix}1+6+2&3+0+4&2-3+6\\2+0-1&6+0-2&4+0-3\\1+4+3&3+0+6&2-2+9\end{bmatrix}$$

$$=\begin{bmatrix}9&7&5\\1&4&1\\8&9&9\end{bmatrix}$$

& $$A^3=A^2\times A=\begin{bmatrix}9&7&5\\1&4&1\\8&9&9\end{bmatrix}\cdot\begin{bmatrix}1&3&2\\2&0&-1\\1&2&3\end{bmatrix}=\begin{bmatrix}28&37&26\\10&5&1\\35&42&34\end{bmatrix}$$

$$\therefore A^3-4A^2-3A+11I$$

$$=\begin{bmatrix}28&37&26\\10&5&1\\35&42&34\end{bmatrix}-4\begin{bmatrix}9&7&5\\1&4&1\\8&9&9\end{bmatrix}-3\begin{bmatrix}1&3&2\\2&0&-1\\1&2&3\end{bmatrix}+11\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

Related Matrices:-

① Transpose of Matrix:

The matrix obtained from any given matrix A, by interchanging rows & columns is called the transpose of A & denoted by A'

Thus Transpose of $A=\begin{bmatrix}1&2\\4&5\\7&8\end{bmatrix}$ is $A'=\begin{bmatrix}1&4&7\\2&5&8\end{bmatrix}$.

& $(A')'=A$ (always).

& in general if $A_{m\times n}$ then $A'_{n\times m}$.

Observations:

$(AB)' = B'A'$

any square matrix $A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A')$.

$\uparrow$ symm $\quad\quad$ $\uparrow$ skewsymm.

Example:- Express the matrix A as the sum of a symmetric & skewsymmetric matrix where

$$A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$$

$\Rightarrow$ as we know

Any A-matrix can be written as

$$A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A').$$

$$\therefore A' = \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix} \quad (\text{from } A).$$

$$\therefore A+A' = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix} + \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & -14 \end{bmatrix}$$

$$\& \; A-A' = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix} - \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$$

$$\therefore A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A')$$

$$= \frac{1}{2}\begin{bmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & -14 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1.5 & -4 \\ 1.5 & 3 & -3 \\ -4 & -3 & -7 \end{bmatrix} + \begin{bmatrix} 0 & 0.5 & 1 \\ -0.5 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

Adjoint of matrix.

The determinant of the square matrix.

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ is } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The matrix formed by the cofactors of the element of Δ is

$$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$. Then the transpose of this matrix.

i.e. $$\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$ is called the adjoint of the matrix A

Thus adj. of A is transposed matrix of cofactor of A.

Inverse of matrix.

If A be any matrix, then matrix B, if exists, such that $AB = BA = I$ is called the inverse of A which is denoted by A^{-1} so that $AA^{-1} = I$.

Also $$A^{-1} = \frac{\text{Adj } A}{|A|} \quad \because |A| \neq 0$$

inverse of matrix is Unique. & $(AB)^{-1} = B^{-1}A^{-1}$.

Example: Find the inverse of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

$\Rightarrow$ Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ say $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Now $$|A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{vmatrix} = 1(-12-12) - 1(-4-6) + 3(-4+6)$$
$$= -24 + 10 + 6 = -8$$

if $A_1, A_2, \cdots$ be cofactors of $a_1, a_2 \cdots$ in A. (7)

then $A_1 = \begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} = -24$

$A_2 = -\begin{vmatrix} 1 & 3 \\ -4 & -4 \end{vmatrix} = -(-4+12) = -8$

$A_3 = \begin{vmatrix} 1 & 3 \\ 3 & -3 \end{vmatrix} = (-3-9) = -12$

$B_1 = -\begin{vmatrix} 1 & -3 \\ -2 & -4 \end{vmatrix} = -(-4-6) = -(-10) = 10.$

$B_2 = \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} = (-4+6) = 2$

$B_3 = -\begin{vmatrix} 1 & 3 \\ 1 & -3 \end{vmatrix} = -(-3-3) = -(-6) = 6$

$C_1 = \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} = (-4+6) = 2$

$C_2 = -\begin{vmatrix} 1 & 1 \\ -2 & -4 \end{vmatrix} = -(-4+2) = -(-2) = 2$

$C_3 = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = (3-1) = 2.$

we know that Adj of A = $[\text{Cofactor matrix}]^T$

$$\text{Cofactor Matrix} = \begin{bmatrix} -24 & 10 & 2 \\ -8 & 2 & 2 \\ -12 & 6 & 2 \end{bmatrix}$$

$$\therefore \text{Adj of } A = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

Hence inverse of the given matrix A.

$$= \frac{\text{Adj}A}{\Delta} = \frac{1}{-8}\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Example: Find Matrix A if $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$

$\Rightarrow$ If $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = B$, $\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = C$ & $\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix} = D$ then

From given,

$BAC = D$ or $AC = B^{-1}D$.

$\therefore A = B^{-1}DC^{-1}$

$\therefore$ Find $B^{-1} = \dfrac{\text{Adj} B}{|B|} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ $\left\{ \because \right.$ Adj of B. $= \frac{1}{1}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}^T$

& $C^{-1} = \dfrac{\text{Adj} C}{|C|} = \begin{bmatrix} 8 & 2 \\ 5 & 3 \end{bmatrix}$ $\left\{ \because \right.$ Adj of C $= \frac{1}{-1}\begin{bmatrix} -3 & -5 \\ -2 & -3 \end{bmatrix}^T$

Hence $A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$.

$$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 14 & 8 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 13 \\ -34 & -18 \end{bmatrix}$$

(1) Rank of Matrix:-

A matrix is said to be of rank r when
(1) It has at least one non-zero minor of order r
(2) Every minor of order higher than r vanishes.

(2) Elementary transformation of a matrix:

The following operations, three of which refer to rows & three to columns are knowns as elementary transformation:
(1) The interchange of any two rows (columns) $\Rightarrow R_{ij}$
(2) The multiplication of any two (rows)(columns) by non-zero numbers. $\Rightarrow KR_i$

③ The addition of const multiple of elements of any row (column) to corresponding elements of any other row (column) ⇒ $(R_i + pR_j)$ (9)

Equivalent Matrix :- (~).

A & B are equivalent if
one can be obtained from the other by seq. of elementary transformations

Example: Determine the rank of the following matrices:

① $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

To find minors, lets convert it into more simple form.

i.e. operate $R_2 - R_1$ & $R_3 - 2R_1$.

$\therefore A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$

as we know: To find rank, we want to find highest order non-zero minor.

$\therefore$ first find minor of 3×3 order., by C_1

$|A| = 1(-2+2) - 0(\;) + 0(0) = 0.$

$\therefore$ ~~3×3~~ order three minor is Zero

$\therefore$ rank of matrix is not 3

similarly for other also.

& for 2nd order.

$\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \neq 0$ also $\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} \neq 0$ but $\begin{vmatrix} 0 & 2 \\ 0 & 2 \end{vmatrix} = 0$, $\begin{vmatrix} 2 & -1 \\ 2 & -1 \end{vmatrix} = 0$

but there exist at two minors of order 2 are not zero.

$\therefore$ Rank of matrix = 2.

i.e. $\boxed{\rho(A) = 2}$

Practical

(2) $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

make it more simple.

operating $C_3 - C_1,\ C_4 - C_1$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix}$$

$R_3 - R_1,\ R_4 - R_1$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 - 3R_2,\ R_4 - R_2$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 + 3C_2,\ C_4 + C_2$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

obviously 4^{th} order minor is zero

as well as third minor is also zero.

and second order minor $\neq 0$ i.e $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \neq 0$.

$\therefore \exists$ atleast one minor of order 2 is non-zero

$\therefore \boxed{\rho(A) = 2}$

Elementary matrices:

An elementary matrix is that, which obtained from a unit matrix, by subjecting it to any of the elementary transformations.

Example of elementary matrices obtained from.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ are } R_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = C_{23};$$

$$kR_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \ldots \text{ so on.}$$

Gauss-Jordan method of finding the inverse:

Those elementary row transformation which reduces a given square matrix A to the unit matrix, when applied to unit matrix I give the inverse of A.

Working Rule to evaluate:-

Write the two matrices A & I side by side. Then perform the same row transformations on both. As soon as A is reduces to I, the other matrix represents A^{-1}.

Example: Using Guass Jordan method, find the inverse of the matrix. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

$\Longrightarrow$ Writing the same matrix side by side with unit matrix of order 3, we have.

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 1 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

A I

Now we have to convert matrix A to idnit matrix, Apply elementary transformation. & same operatn will be apply on matrix I

∴ Oper $R_2 - R_1$ & $R_3 + 2R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

Now, operate $\frac{1}{2}R_2$ & $\frac{1}{2}R_3$.

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 1 & 0 & \frac{1}{2} \end{array}\right]$$

$R_1 - R_2$, $R_3 + R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -2 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

operate $R_1 + 3R_3$, $R_2 - \frac{3}{2}R_3$ & $(-\frac{1}{2})R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & -5/4 & -1/4 & -3/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right]$$

Hence the inverse of the given matrix is

$$\begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

Example: Use Gauss-Jordan method to find, the inverse of the following matrices.

(i) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

$$\sim \left[\begin{array}{ccc|ccc} 8 & 4 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_1 - 7R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -10 & -4 & 1 & 0 & -7 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 - 2R_1,\ R_3 - R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -10 & -4 & 1 & 0 & -7 \\ 0 & 21 & 9 & -2 & 1 & 14 \\ 0 & 12 & 5 & -1 & 0 & 8 \end{array}\right]$$

$\frac{1}{21} R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -10 & -4 & 1 & 0 & -7 \\ 0 & 1 & 9/21 & -2/21 & 1/21 & 14/21 \\ 0 & 12 & 5 & -1 & 0 & 8 \end{array}\right]$$

$R_3 - 12R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -10 & -4 & 1 & 0 & -7 \\ 0 & 1 & 9/21 & -2/21 & 1/21 & 14/21 \\ 0 & 0 & -1/7 & 1/7 & \frac{-4}{7} & 0 \end{array}\right]$$

$R_1 + 10R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2/7 & 1/21 & 10/21 & -1/3 \\ 0 & 1 & 9/21 & -2/21 & 1/21 & 14/21 \\ 0 & 0 & -1/7 & 1/7 & -4/7 & 0 \end{array}\right]$$

$-7R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2/7 & 1/21 & 10/21 & -1/3 \\ 0 & 1 & 9/21 & -2/21 & 1/21 & 14/21 \\ 0 & 0 & 1 & -1 & -4 & 0 \end{array}\right]$$

$R_2 - \frac{9}{21} R_3,\ R_1 - 2/7 R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 34/21 & -1/3 \\ 0 & 1 & 0 & 1/3 & 37/21 & 14/21 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Normal Form of Matrix :-

Every non-zero matrix A of rank r, can be reduced by a sequence of elementary transformations, to the form.

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ called normal form of A.} \quad —(1)$$

* The rank of a matrix A is r iff. it can be reduced to the normal form (1)
* each elementary matrix is non-singular, therefore

Corresponding to every matrix A of rank r, ∃ non-singular matrices P & Q s.t PAQ equals (1)

If A be a $m \times n$ matrix, then P & Q are square matrices of order m & n respectively.

Row & column operations are allowed, ~~since~~

Example: Reduce the following matrix into its normal form & hence find its rank.

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

To make it normal i.e $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Convert first element to 1.

∴ interchange R_{12}.

$$\therefore A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

operate, $R_2 - 2R_1$, $R_3 - 3R_1$, $R_4 - 6R_1$, $a_{11}, a_{21}, a_{31}, a_{41} = 0$.

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$\therefore$ Operate, C_2+C_1, C_3+2C_1, C_4+4C_1.
To make $a_{12}, a_{13}, a_{14}=0$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_4-R_2-R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

R_2-R_3 (To make a_{22} position 1)

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

R_3-4R_2 (To make $a_{31}=0$)

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

C_3+6C_2, C_4+3C_2 (To make $a_{23}, a_{24}=0$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\frac{1}{33} C_3.$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 - 22C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence $\rho(A) = 3$.

Q. Find non-singular matrices P & Q such that PAQ is in the normal form. Hence find Rank of A.

For matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

$\Rightarrow$ We write.

$A = IAI$

i.e $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{pre-matrix}} A \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{post matrix.}}$

Method of elementary transformations:

We convert Left side matrix to normal form & same operations applies in R.H.S.

but,

When we apply column operation then ~~Change is~~ same operatⁿ applied in RHS only for post matrix.

& When we apply row operatⁿ then same operation applied in R.H.S, only for pre matrix.

∴ To convert first L.H.S in Normal form,

$C_2 - C_1,\ C_3 - 2C_1$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{as it is}} A \underbrace{\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{change.}}$

operate

$R_2 - R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

change   as it is

operate $C_3 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

as it is   change.

$R_3 + R_2$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Change.   as it is

Now L.H.S is in Normal form i.e matrix is converted in Normal form.

& i.e $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ Rank of matrix $= \rho(A) = 2$

& R.H.S is of the form PAQ & both P & Q are Non singular.

$$\therefore \quad P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: Determine Rank of $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix}$ ~~by~~

~~both methods i.e by~~ by Normal form.

* Partition Method of finding inverse :-

if the inverse of A_n ~~of~~ order n is known, then the inverse of A_{n+1} can be easily obtained by adding $(n+1)$th row & column to A_n.

i.e

Let $A = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3' & \alpha \end{array}\right]$ & $A^{-1} = \left[\begin{array}{c|c} X_1 & X_2 \\ \hline X_3' & x \end{array}\right]$.

where A_2, X_2 are column vectors & A_3', X_3' are row vectors (being transpose of column vectors A_3, X_3) &

α, x are ordinary numbers.

We also assume that A_1^{-1} is known.

Then, $AA^{-1} = I_{n+1}$ i.e

$$\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3' & \alpha \end{array}\right]\left[\begin{array}{c|c} X_1 & X_2 \\ \hline X_3' & x \end{array}\right] = \begin{bmatrix} I_n & 0 \\ 0 & 1 \end{bmatrix}$$

it gives,

$$A_1X_1 + A_2X_3' = I_n \quad \text{--- (1)}$$

$$A_1X_2 + A_2x = 0 \quad \text{--- (2)}$$

$$A_3'X_1 + \alpha x_3' = 0 \quad \text{--- (3)}$$

$$A_3'X_2 + \alpha x = 1 \quad \text{--- (4)}$$

From eqn (2) $\boxed{X_2 = -A_1^{-1} A_2 x}$

& from eqn (4) $~~x~~ = \alpha$.

$$A_3'(A_1^{-1}A_2 x) + \alpha x = 1$$

$$x[A_3'(-A_1^{-1}A_2) + \alpha] = 1$$

$$x[\alpha - A_3' A_1^{-1} A_2] = 1$$

$$\therefore x = \frac{1}{(\alpha - A_3' A_1^{-1} A_2)}$$

$$\boxed{x = (\alpha - A_3' A_1^{-1} A_2)^{-1}.}$$

Also from (1)

$$X_1 = A_1^{-1}(I_n - A_2 X_3').$$

& using this, (3) becomes.

$$X_3' = -A_3' A_1^{-1}(\alpha - A_3' A_1^{-1} A_2)^{-1}$$

$$= -A_3' A_1^{-1} x.$$

Then X_1 is determined & hence A^{-1} is computed

Example:- Using the partition method, find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$.

$\Rightarrow$

$$A = \left[\begin{array}{cc:c} 1 & 1 & 1 \\ 4 & 3 & -1 \\ \hdashline 3 & 5 & 3 \end{array}\right] = \left[\begin{array}{c:c} A_1 & A_2 \\ \hdashline A_3' & \alpha \end{array}\right]$$

we should know A_1^{-1}

$$A_1 = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix} \text{ then } A_1^{-1} = \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix}$$

Let $A^{-1} \left[\begin{array}{c|c} X_1 & X_2 \\ \hline X_3' & x \end{array}\right]$ so that $AA^{-1} = I$.

To find x,

$$x = (\alpha - A_3' A_1^{-1} A_2)^{-1} \quad \text{—} ①$$

as $\alpha = 3$,

$A_3' = [3 \; 5]$, $A_1^{-1} = \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$\therefore A_3' A_1^{-1} A_2 = -[3 \; 5] \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= -\begin{bmatrix} 9-20 & -3+5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= -[-11 \quad 2] \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= -[-11 - 2]$$

put in ① $= -[-13] = [13]$

$$\therefore x = (3 - (13))^{-1} = -10.$$

$$\therefore x = (-10)^{-1}$$

$$\boxed{x = -\frac{1}{10}}$$

$$X_2 = -A_1^{-1} A_2 x$$

$$= -\left[-\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \right] \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left(-\frac{1}{10}\right)$$

$$= \begin{bmatrix} 3+\cancel{2} \\ -4-1 \end{bmatrix} \left(-\frac{1}{10}\right).$$

$$= \begin{bmatrix} \cancel{5}4 \\ -5 \end{bmatrix} \left[-\left(\frac{1}{10}\right)\right] = -\frac{1}{10} \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

Then $X_3' = -A_3' A_1^{-1} \alpha$

$$= -[3, 5]\left(-\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix}\right)\left(-\frac{1}{10}\right)$$

$$= [9-20 \quad -3+5]\left(-\frac{1}{10}\right)$$

$$= [-11 \quad 2]\left(-\frac{1}{10}\right)$$

$$= \left(-\frac{1}{10}\right)[-11, 2].$$

& $X_1 = A_1^{-1}(I - A_2 X_3') = A_1^{-1} - A_1^{-1} A_2 X_3'$

$$= -\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}[-11 \quad 2].$$

$$= -\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} + \begin{bmatrix} 3+1 \\ -4-1 \end{bmatrix}[-11 \quad 2]$$

$$= -\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} + \begin{bmatrix} 4 \\ -5 \end{bmatrix}[-11 \quad 2]\left(-\frac{1}{10}\right)$$

$$= -\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} + \begin{bmatrix} -44 & 8 \\ 55 & -10 \end{bmatrix}\left(-\frac{1}{10}\right)$$

$$= \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix} + \begin{bmatrix} -44 & 8 \\ 55 & -10 \end{bmatrix}\left(-\frac{1}{10}\right)$$

$$= \cancel{\begin{bmatrix} -47 & -7 \\ -51 & -11 \end{bmatrix}}$$

$$= \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} -4.4 & 0.8 \\ 5.5 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.4 & 0.2 \\ -1.5 & 0 \end{bmatrix}$$

$$\therefore A^{-1} = \left[\begin{array}{c|c} X_1 & X_2 \\ \hline X_3' & \alpha \end{array}\right] = \begin{bmatrix} 1.4 & 0.2 & -0.4 \\ -1.5 & 0 & 0.5 \\ 1.1 & -0.2 & -0.1 \end{bmatrix}$$

Tw

Find the inverse of matrix by using partition method $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$.

* Solution of Linear System of Equations:

① Method of determinants - Cramer's Rule:

Consider the equations $\left\{\begin{array}{l} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{array}\right\} \cdots ①$

if the determinant of coeff. be

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = D. \quad (say).$$

$$\text{then } x = \frac{D_x}{D}, \quad y = \frac{D_y}{D} \quad \& \quad z = \frac{D_z}{D}.$$

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \quad D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

② Matrix inversion Method:

Consider equations.

$$\left.\begin{array}{l} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3. \end{array}\right\} - ①.$$

If A is coeff matrix of x, y, z, X is matrix of unknowns x, y, z & D is matrix of constants, which is in R.H.S.

$\therefore$ eqⁿ ① equivalent to the matrix.

$AX = D$. —— ②

where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \; \& \; D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$\therefore$ if we multiply eqⁿ ② by A^{-1} on both sides, we get.

$$A^{-1}AX = A^{-1}D.$$

$$IX = A^{-1}D$$

$$\therefore \boxed{X = A^{-1}D.}$$

Example: Solve the equations $3x+y+2z=3$, $2x-3y-z=-3$, $x+2y+z=4$ by ① determinants, ⓘⓘ by matrix.

$\Longrightarrow$ ① by determinant method.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = D.$$

$$\therefore D = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3(-3+2)-1(2+1)+2(4+3)$$
$$= 3(-1)-1(3)+2(7)$$
$$= -3-3+14 = 8$$

$$\therefore \boxed{D = 8}.$$

Now

$$Dx = \begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix} = 3(-3+2)-1(-3+4)+2(-6+12)$$
$$= 3(-1)-1(1)+2(+6)$$
$$= -3-1+12 = 8$$

$\therefore \boxed{Dx = 8}$

$$D_y = \begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix} = 3(-3+4) - 3(2+1) + 2(8+3)$$
$$= 3(1) - 3(3) + 2(11)$$
$$= 3 - 9 + 22 = +16$$

$\therefore \boxed{D_y = 16}$

$$\& \ D_z = \begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix} = 3(-12+6) - 1(8+3) + 3(4+3)$$
$$= 3(-6) - 1(11) + 3(7)$$
$$= -18 - 11 + 21 = -8$$

$\therefore D_z = -8.$

$\therefore$ by Cramers Rule.

$$x = \frac{Dx}{D} = \frac{8}{8} = 1 \Rightarrow \boxed{x = 1}$$

$$y = \frac{Dy}{D} = \frac{16}{8} = 2 \Rightarrow \boxed{y = 2}$$

$$\& \ z = \frac{Dz}{D} = \frac{-8}{8} = -1 \Rightarrow \boxed{z = -1.}$$

(ii) By matrix method:

A is coeff matrix i.e $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ \& \ D = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$

$\therefore X = A^{-1}D.$

To find A^{-1}, (by Adjoint method).

$$A^{-1} = \frac{\text{Adj } A}{\Delta}$$

To find Adjoint of A.

$$\text{Cofactor matrix} = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -3 & 7 \\ 3 & 1 & -5 \\ 5 & 7 & -11 \end{bmatrix}$$

Now Adj = Transpose of Cofactor matrix

$$= \begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{Adj}A}{|A|} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -3-9+20 \\ -9-3+28 \\ 21+15-44 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 8 \\ 16 \\ -8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \Rightarrow \boxed{x=1}, \boxed{y=2} \; \& \; \boxed{z=-1}$$

Example: Solve the equations

$x_1 - x_2 + x_3 + x_4 = 2$;

$x_1 + x_2 - x_3 + x_4 = -4$;

$x_1 + x_2 + x_3 - x_4 = 4$;

$x_1 + x_2 + x_3 + x_4 = 0$

by finding the inverse by elementary row operations.

$x_1 = 1$
$x_2 = -1$
$x_3 = -2$
$x_4 = -2$

Consistency of Linear system of Equations

Consider the system of equations.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= k_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= k_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= k_m \end{aligned}\right\} \text{—— (1)}$$

containing n-Unknowns. $x_1, x_2, \ldots, x_n$

To determine whether equations (1) are consistant (i.e posses solution) or not. we consider the rank of matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \; \& \; K = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & k_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & k_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & k_n \end{bmatrix}$$

$\boxed{AX = K}$

where, A = coeff. matrix.

K = Augmented matrix of eqn (1).

Rouche's Thm :- The system of equations (1) is consistent if the coeff matrix A & Augmented matrix K are of the same rank otherwise the system is inconsistent

Procedure to test the consistency of a system of equations in n unknowns:

Find the ranks of the coeff. matrix & augmented matrix K, by reducing A to triangular form by elementary row operations.

Let rank of A is r &

rank of K is r'

then. $\rho(A)$ = no. of Row − no. of zero rows.

① If $r \neq r'$, the equations are inconsistent i.e, there is no solution.

② If $r = r' = n$ (= no. of unknowns) system is consistent & has unique soln.

③ If $r = r' < n$ (no. of Unknowns). system is inconsistent & has infinitely many solutions.

* Giving arbitrary values to n−r of unknowns we may express the other r-unknowns in terms of these.

Test the consistency of equations & solve.

$5x + 3y + 7z = 4$
$3x + 26y + 2z = 9$
$7x + 2y + 10z = 5$

⟹ write in Form, $Ax = K$.

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

A B

write Augumented matrix. i.e [A:K]

$$\begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$$

by using elementary row transformations convert it into triangular form.

operate, $3R_1, 5R_2$

$$\begin{bmatrix} 15 & 9 & 21 & 12 \\ 15 & 130 & 10 & 45 \\ 7 & 2 & 10 & 5 \end{bmatrix}$$

$R_2 - R_1$

$$\begin{bmatrix} 15 & 9 & 21 & 12 \\ 0 & 121 & -11 & 33 \\ 7 & 2 & 10 & 5 \end{bmatrix}$$

$\frac{R_1}{3}, \frac{R_2}{11},$

$$\begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 11 & -1 & 3 \\ 7 & 2 & 10 & 5 \end{bmatrix}$$

$7R_1, 5R_3$

$$\begin{bmatrix} 35 & 21 & 49 & 28 \\ 0 & 11 & -1 & 3 \\ 35 & 10 & 50 & 25 \end{bmatrix}$$

$R_3 - R_1$

$$\begin{bmatrix} 35 & 21 & 49 & 28 \\ 0 & 11 & -1 & 3 \\ 0 & -11 & -1 & -3 \end{bmatrix}$$

$R_3 + R_2$

$$\left[\begin{array}{ccc|c} 35 & 21 & 49 & 28 \\ 0 & 11 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \frac{R_1}{7} \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 11 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

above matrix is upper triangular.

where rank of A = 2 & rank of Augmented matrix AK = 2

$\therefore \; r = r' < n$

i.e $r = r' < 3$ (no. of unknowns).

System is consistent has infinitely many soln.

i.e. $\begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$

$\therefore \; 5x + 3y + 7z = 4$

$11y - z = 3$

& z is parameter.

$\therefore \; y = \frac{3 + z}{11}$

$= \frac{3}{11} + \frac{z}{11}$

& $5x = 4 - 3y - 7z$

$x = \frac{4}{5} - \frac{3}{5}\left(\frac{3}{11} + \frac{z}{11}\right) - \frac{7}{5}z$

$= \frac{4}{5} - \frac{9}{55} - \frac{3z}{55} - \frac{77z}{55}$

$= \frac{44-9}{55} + \left(\frac{-3z - 77z}{55}\right)$

$= \frac{35}{55} + \left(\frac{-80}{55}z\right).$

$x = \frac{7}{11} - \frac{16}{11}z$

z is parameter,

Hence $x = \frac{7}{11}$, $y = \frac{3}{11}$ & $z = 0$ is particular solution.

Example: Examine for the consistency & if consistent Solve the system.

$x+y+z=3$; $x+2y+3z=4$; $4+4y+9z=6$.

$y=1$, $z=0$

System of linear homogeneous Equations.

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=0 \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=0 \\ \vdots \\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=0. \end{array}\right\} \text{---} (1)$$

Find Rank of matrix A by using elementary transformation,

(1) If $r=n$ then eq^n (1) have trivial sol^n i.e

$x_1=x_2=\cdots x_n=0$

(2) If $r<n$, then eq^n (1) $n-r$ linearly independent sol^n.

i.e infinitely many solut^n.

(3) m = no. of eq^{n}s.

n = no. of unknowns.

$m<n$:

then it has infinitely many sol^n.

(other than $x_1=x_2\cdots=x_n=0$.

(4) $m=n$ it has non-trivial solution.

Example: $x+2y+3z=0$; $3x+4y+4z=0$,

$7x+10y+12z=0$.

$\Rightarrow$

$$A = \text{coeff matrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$$

R_2-3R_1

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 7 & 10 & 12 \end{bmatrix}$$

$R_3 - 7R_1 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix}$

$R_3 - 2R_2 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$

it is upper triangular.

$\therefore \rho(A) = 3$.

i.e $\boxed{r = n}$

$\therefore$ it has trivial solution.

$x = y = z = 0$.

(ii) $4x + 2y + z + 3w = 0$

$6x + 3y + 4z + 7w = 0$

$2x + y + w = 0$.

coeff. matrix.

$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

$\frac{R_1}{4} \sim \begin{bmatrix} 1 & 1/2 & 1/4 & 3/4 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

$4 - 6/4 = 4 - \frac{3}{2} = \frac{8-3}{2} = 5/2$

$7 - \frac{18}{4} = 7 - \frac{9}{2}$

$\frac{14-9}{2} = 5/2$

$R_2 - 6R_1$

$\sim \begin{bmatrix} 1 & 1/2 & 1/4 & 3/4 \\ 0 & 0 & 5/2 & 5/2 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

$1 - 6/4 = 1 - 3/2 = -1/2$

$R_3 - 2R_1 \sim \begin{bmatrix} 1 & 1/2 & 1/4 & 3/4 \\ 0 & 0 & 5/2 & 5/2 \\ 0 & 0 & -1/2 & -1/2 \end{bmatrix}$

$2R_2, 2R_3 \sim$

$$\begin{bmatrix} 1 & 1/2 & 1/4 & 3/4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$R_3 + \frac{1}{5}R_2$

$$\sim \begin{bmatrix} 1 & 1/2 & 1/4 & 3/4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\sim}{4R_1} \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r = 2$ i.e $\rho(A) = 2. = 3 - 1 = 2$

$\therefore$ it has infinitely many soln.

$4x + 2y + z + 3w = 0$ — (1)

& $z + w = 0.$ — (2)

$m = 4$ & $r = 2.$

$\therefore$ No. of independent soln $= 4 - 2 = 2.$

from (2). $\boxed{z = -w}$.

$\therefore$ From (1).

$4x + 2y + (-w) + 3w = 0$

$4x + 2y + 2w = 0$

$\Rightarrow 2y = -4x - 2w$

$\boxed{y = -2x - w}$

x & w are parameters.

Q. Solve the system of equations.

$x + 2y + 3z = 0;$

$2x + 3y + z = 0;$

$4x + 5y + 4z = 0;$

$x + y - 2z = 0.$

Example :-

Determine the value of λ for which the equations $3x_1+2x_2+4x_3=3$; $x_1+x_2+x_3=\lambda$; $5x_1+4x_2+6x_3=15$ are consistent. Find also the corresponding solution :

$\Rightarrow$ Given system of equations,

$$3x_1+2x_2+4x_3=3$$
$$x_1+x_2+x_3=\lambda$$
$$5x_1+4x_2+6x_3=15$$

it can be written in matrix form as,

$AX = B.$

$$A=\begin{bmatrix}3&2&4\\1&1&1\\5&4&6\end{bmatrix} = \text{coeff matrix.}$$

$$X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \quad \& \quad B=\begin{bmatrix}3\\\lambda\\15\end{bmatrix}$$

Now to check consistency : Find $\rho(A)$ & $\rho(A:B)$.

Consider,

$$[A:B]=\left[\begin{array}{ccc:c}3&2&4&3\\1&1&1&\lambda\\5&4&6&15\end{array}\right]$$

We have to make it upper triangular.

$\therefore$ To convert a_{11} position 1.

$\therefore$ R_{12} ($R_1 \leftrightarrow R_2$).

$$\sim\left[\begin{array}{ccc:c}1&1&1&\lambda\\3&2&4&3\\5&4&6&15\end{array}\right]$$

R_2-3R_1, R_3-5R_1 (To obtain all zero below a_{11}).

$$\sim\left[\begin{array}{ccc:c}1&1&1&\lambda\\0&-1&1&3-3\lambda\\0&-1&1&15-5\lambda\end{array}\right]$$

$R_3 - R_2$ (To obtain zeros below a_{22}.

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & \lambda \\ 0 & -1 & 1 & 3-3\lambda \\ 0 & 0 & 0 & 12-2\lambda \end{array}\right]$$

Now we know that, system is consistent.

i.e $\rho(A) = \rho(A:B)$

it is possible only if $12 - 2\lambda = 0$

$\Rightarrow 12 = 2\lambda$

$\Rightarrow \boxed{\lambda = 6}$

Example: ② For different values of k, discuss the following equations:

$x + 2y - z = 0$;

$3x + (k+7)y - 3z = 0$

$2x + 4y + (k-3)z = 0$.

$\Rightarrow \quad Ax = Z.$

$\therefore$ [$Ax = Z$] is homogeneous system.

where, $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & k+7 & -3 \\ 2 & 4 & k-3 \end{bmatrix}$ = coeff matrix.

$[A:Z]$ = Augumented matrix.

To check consistency, find rank of A.

consider

$R_2 - 3R_1$ & $R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & k+1 & 0 \\ 0 & 0 & k-1 \end{bmatrix}$$

Now, to discuss different condⁿ.

① System have infinite solⁿ if.

$r < n$ i.e $\rho(A) < 3$.

it is possible only if

$k+1 = 0$ or $k-1 = 0$

i.e $k = -1$ or $k = 1$.

① For $k = 1$.

$$[A|Z] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ (from final solⁿ)

Matrix gives eqⁿ.

$x + 2y - z = 0$

& $2y = 0 \Rightarrow \boxed{y = 0}$

but z is parameter.

$\therefore \boxed{x = z}$

② For $k = -1$.

$$[A|Z] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$x + 2y - z = 0$

$-2z = 0 \Rightarrow \boxed{z = 0}$

y is parameter.

$\boxed{x = -2y}$

& system has uniques solⁿ if $k \neq \pm 1$.